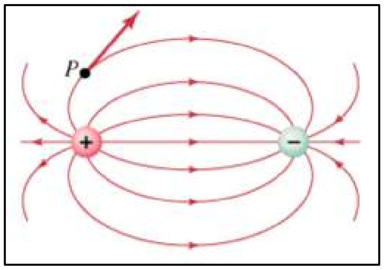
Test 4 - Electricity



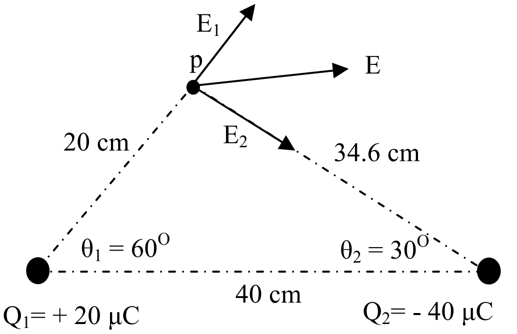
Electric Fields – An electric field is a representation of the effects that electric charges have on a region of space. If I place an electric charge in the vicinity of other charges it will be pushed or pulled in certain directions. These lines of force make up the electric field.

In the diagram at the right a positive charge and a negative charge are in a region of space. The small positive test charge "p" is brought into the region and will be simultaneously repelled by the positive charge and attracted to the negative charge, and will follow the lines shown depending on its starting point.

- **Determine electric field for an electric dipole.**
  The diagram above shows an electric dipole – one positive charge and one negative charge. Since the other book shows us that $E = \frac{F}{q} = \frac{\frac{kQq}{r^2}}{q} = \frac{kQ}{r^2}$ for a single charge, we can use the "principle of superposition" to add the effects of multiple charges. The only real trick to it is to remember that these are vector quantities and that direction matters. To find the strength of the electric field at any point, say point p above, we need to add the effects of electric fields from the other two charges.

$$E = \frac{kQ_1}{r_1^{\,2}} + \frac{kQ_2}{r_2^{\,2}}$$

Example 1: Find the electric field at point p due to Q_1 and Q_2



$$E_1 = \frac{kQ_1}{r_1^{\,2}} = \frac{(9\times10^9)(20\mu C)}{(0.2m)^2} = 4.5\times10^6\,\frac{N}{C}\angle 60^o$$

$$= 2.25\times10^6\hat{i} + 3.89\times10^6\hat{j}$$

$$E_2 = \frac{kQ_2}{r_2^{\,2}} = \frac{(9\times10^9)(40\mu C)}{(0.346m)^2} = 3.0\times10^6\,\frac{N}{C}\angle 330^o$$

$$= 2.60\times10^6\hat{i} - 1.50\times10^6\hat{j}$$

$$E = (2.25\times10^6 + 2.60\times10^6)\hat{i} + (3.89\times10^6 - 1.50\times10^6)\hat{j}$$

$$= 4.85\times10^6\hat{i} + 2.39\times10^6\hat{j}$$

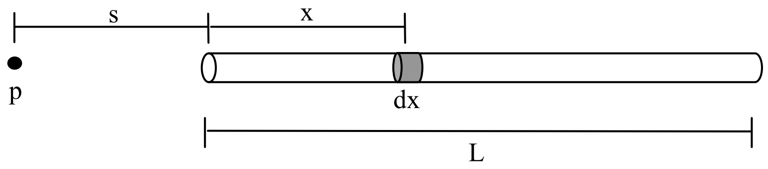
$$= 5.41\times10^6\angle 26.2^o$$

This works fine for discrete charges, but quickly becomes unmanageable with continuously charged objects. For example, imagine a charge distributed uniformly along the length of a rod or over a disc or other shaped object. In these kinds of cases, we need to add up the effects of each little piece of charge with calculus.

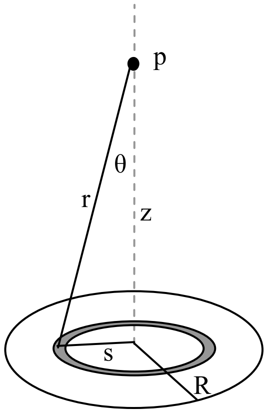
These are the steps to follow:

Steps	**For Q distributed over a length**	**For Q distributed over an area**	**For Q distributed over a volume**
1) Define a charge distribution	$\lambda = Q/L$	$\sigma = Q/A$	
2) Find dQ, the charge on a infinitesimal slice of the object	$dQ = \lambda\ dx$	$dQ = \lambda\ dA$	
3) Find r, the distance from p to dx	will vary – use geometry		
4) Integrate the effects of each slice of charge	$E = \int \frac{k\,dQ}{r^2}$		

Example 2: Find the electric field at point p a distance of s from one end of a rod of length L with charge Q



Steps	**For Q distributed over a length**
1) Define a charge distribution	$\lambda = Q/L$
2) Find dQ, the charge on a infinitesimal slice of the object	$dQ = \lambda\ dx$
3) Find r, the distance from p to dx	$r = s + x$
4) Integrate the effects of each slice of charge	$E = \int \frac{k\,dQ}{r^2}$ $= \int \frac{k\lambda}{(s+x)^2}\,dx$ $= k\lambda \int \frac{dx}{(s+x)^2}$ $= k\lambda \int \frac{dx}{(s+x)^2}$ $= k\lambda \int (s+x)^{-2}\,dx$ $= -k\lambda\left[(s+x)^{-1} \Big\|_{x=0}^{x=L} \right]$ $= -k\lambda\left[\frac{1}{(s+x)} \Big\|_{x=0}^{x=L} \right]$ $= -k\lambda\left[\frac{1}{s+L} - \frac{1}{s+0} \right]$ $= -k\lambda\left[\frac{s}{s(s+L)} - \frac{s+L}{s(s+L)} \right]$ $= -k\lambda\left[\frac{s-s-L}{s(s+L)} \right]$ $= -k\lambda\left[\frac{-L}{s(s+L)} \right]$ $= -k\frac{Q}{L}\left[\frac{-L}{s(s+L)} \right]$ $= \frac{kQ}{s(s+L)}$



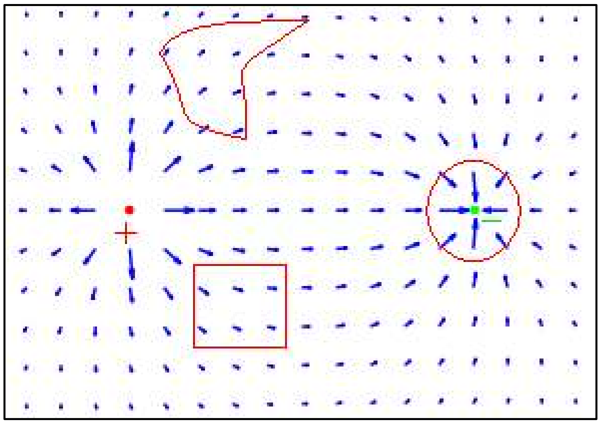
Example 3: A flat disk of radius R has charge Q. Find the electric field at point p on the axis at a distance of z.

Here we want to imagine cutting the disc up into concentric rings. The area of each of these rings is dA = 2πs ds. Since the rings are symmetric around the axis, all horizontal components of the electric field will cancel out. The remaining vertical component, $E_Z = E \cos\theta = E\,(z/r)$.

We'll also need to do a "U substitution", where $U = z^2 + S^2$ and $dU = 2s\,ds$.

Steps	For Q distributed over an area
1) Define charge distribution	$\sigma = Q / A = Q / \pi R^2$
2) Find dQ	$dQ = \sigma\, dA = \sigma\, 2\pi s\, ds$
3) Find r	$r = \sqrt{z^2 + s^2}$
4) Integrate	$E = \int \frac{k\,dQ}{r^2}\cos\theta$ $= \int \frac{k\sigma\, 2\pi s}{\left(\sqrt{z^2+s^2}\right)^2}\left(\frac{z}{r}\right) ds$ $= \int \frac{k\sigma\, 2\pi s}{\left(\sqrt{z^2+s^2}\right)^2}\left(\frac{z}{\sqrt{z^2+s^2}}\right) ds$ $= 2\pi k\sigma\, z \int \frac{s}{\left(z^2+s^2\right)^{\frac{3}{2}}}\, ds$ $= 2\pi k\sigma\, z \int \frac{\frac{1}{2}\,dU}{U^{\frac{3}{2}}}$ $= \pi k\sigma\, z \left[-\frac{1}{U^{\frac{1}{2}}} \right]$ $= -\pi k\sigma\, z \left[\frac{1}{\sqrt{z^2+s^2}} \right] \Bigg|_{s=0}^{s=R}$ $= -\pi k\sigma\, z \left[\frac{1}{\sqrt{z^2+s^2}} \right] \Bigg|_{s=0}^{s=R}$ $= -\pi k\sigma z \left[\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{\sqrt{z^2}} \right]$ $= -\frac{\pi k z Q}{\pi r^2} \left[\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{\sqrt{z^2}} \right]$ $= -\frac{k z Q}{r^2} \left[\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{\sqrt{z^2}} \right]$

- **Apply Gauss' Law to determine electric field.**



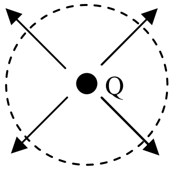
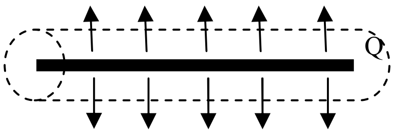
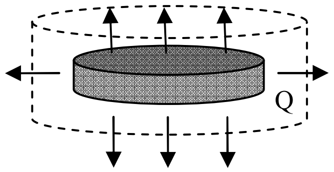
In a given electric field, we can imagine any number of closed surfaces, like those shown to the right. Depending on which surface we draw there are electric field lines flowing into and/or out of that surface. The lines are called electric flux lines and are represented by Φ. Looking at the square, two flux lines flow into the surface and two flow out. Considering the circle, eight flux lines flow in, and none flow out.

Gauss's law gives the equivalence relation between any flux flowing in/out of any closed surface and the result of inner sources and sinks, such as electric charges, within the closed surface.

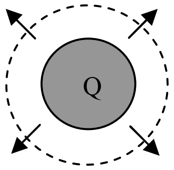
In its integral form, the law states $\Phi = \oint_S E \cdot dA = \frac{Q_{enclosed}}{\varepsilon_O}$

where Q is the charge enclosed by the closed surface and ε_O is the permittivity of free space = $8.85 \times 10^{-12}\ C^2 / Nm^2$. The circle on the integration symbol indicates performing integration on the surface we have drawn enclosing some volume, and dA is a differential area on the closed surface *S*.

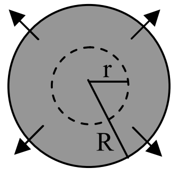
Gauss' Law is easier than the first method of integrating to find the electric field, but you must carefully choose what surface to imagine around the charge. You want the electric field to enter/exit the surface at right angles so that symmetry makes the integration easy.

If the charge is:	then pick:	Like this:
Spherical	Spherical	
Long cylinder	Long cylinder	
Planar	Flat cylinder	

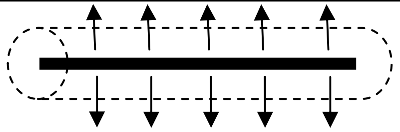
Example 4: Find the electric field due to a sphere with a charge of Q at a distance of r from the center of the sphere.

Step 1: Imagine a spherical surface surrounding the spherical charge.	
Step 2: Find the charge enclosed by the surface	Q
Step 3: Apply Gauss' law – since E is symmetric it is a constant and can be brought outside of the integral. Then the integral of dA is simply the surface area of the surface.	$\oint_S E \cdot dA = \frac{Q_{enclosed}}{\varepsilon_O}$ $E\oint_S dA = \frac{Q_{enclosed}}{\varepsilon_O}$ $E\,(4\pi\, r^2) = \frac{Q_{enclosed}}{\varepsilon_O}$
Step 4: Solve for E (note – we end up with the original definition of E)	$E = \frac{Q_{enclosed}}{4\pi\, r^2 \varepsilon_0}$ $or \;\; E = \frac{kQ}{r^2}$ $\text{since k} = \frac{1}{4\pi\varepsilon_0}$

Example 5: Find the electric field inside a sphere with a radius of R and a charge distribution of ρ at a distance of r from the center of the sphere.

Step 1: Imagine a spherical surface surrounding the spherical charge.	
Step 2: Find the charge enclosed by the surface	$Q_{enclosed} = \rho\left(\frac{4}{3}\pi r^3\right)$
Step 3: Apply Gauss' law – since E is symmetric it is a constant and can be brought outside of the integral. Then the integral of dA is simply the surface area of the surface.	$\oint_S E \cdot dA = \frac{Q_{enclosed}}{\varepsilon_O}$ $E\oint_S dA = \frac{Q_{enclosed}}{\varepsilon_O}$ $E\,(4\pi r^2) = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\varepsilon_O}$
Step 4: Solve for E	$E = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi r^2 \varepsilon_0}$ $E = \frac{\rho r}{3\varepsilon_0}$

Example 5: Find the electric field due to a wire of radius R and length L and a uniform charge distribution of ρ at a distance of r from the center of the wire.

Step 1: Imagine a cylindrical surface. If r <R the surface is inside the cylinder, r>=R then outside	
Step 2: Find the charge enclosed by the surface	$Q_{enclosed} = \rho\left(\pi r^2 L\right)$
Step 3: Apply Gauss' law – since E is symmetric it is a constant and can be brought outside of the integral. Then the integral of dA is simply the surface area of the surface.	$\oint_S E \cdot dA = \frac{Q_{enclosed}}{\varepsilon_O}$ $E\oint_S dA = \frac{Q_{enclosed}}{\varepsilon_O}$ $E\,(2\pi rL) = \frac{\rho\left(\pi r^2 L\right)}{\varepsilon_O}$
Step 4: Solve for E	$E = \frac{\rho r}{2\,\varepsilon_0}$